GRADED ANNIHILATORS AND TIGHT CLOSURE TEST IDEALS

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Abstract. Let R be a commutative Noetherian local ring of prime characteristic p, with maximal ideal $\mathfrak{m}$. The main purposes of this paper are to show that if the injective envelope E of $R/\mathfrak{m}$ has a structure as an x-torsion-free left module over the Frobenius skew polynomial ring over R (in the indeterminate x), then R has a tight closure test element (for modules) and is F-pure, and to relate the test ideal of R to the smallest 'E-special' ideal of R of positive height.

A byproduct is an analogue of a result of Janet Cowden Vassilev: she showed, in the case where R is an F-pure homomorphic image of an F-finite regular local ring, that there exists a strictly ascending chain $0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_t = R$ of radical ideals of R such that, for each $i = 0, \ldots, t-1$, the reduced local ring R/τ_i is F-pure and its test ideal (has positive height and) is exactly τ_{i+1}/τ_i. This paper presents an analogous result in the case where R is complete (but not necessarily F-finite) and E has a structure as an x-torsion-free left module over the Frobenius skew polynomial ring. Whereas Cowden Vassilev's results were based on R. Fedder's criterion for F-purity, the arguments in this paper are based on the author's work on graded annihilators of left modules over the Frobenius skew polynomial ring.

0. Introduction

This paper presents new information about test elements in tight closure theory in commutative algebra.

Throughout the paper, R will denote a commutative Noetherian ring of prime characteristic p. We shall always denote by $f : R \longrightarrow R$ the Frobenius homomorphism, for which $f(r) = r^p$ for all $r \in R$. This paper will make use of the author's work in [14] on left modules over the *Frobenius skew polynomial ring over* R, that is, the skew polynomial ring $R[x, f]$ associated to R and f in the indeterminate x over R. Recall that $R[x, f]$ is, as a left R-module, freely generated by $(x^i)_{i\in\mathbb{N}_0}$ (I use $\mathbb{N}$ and $\mathbb{N}_0$ to denote the set of positive integers and the set of non-negative integers, respectively), and so consists of all polynomials $\sum_{i=0}^{n} r_i x^i$, where $n \in \mathbb{N}_0$ and $r_0, \ldots, r_n \in R$; however, its multiplication is subject to the rule $xr = f(r)x = r^p x$ for all $r \in R$. Note that $R[x, f]$ can be considered as a positively-graded ring $R[x, f] = \bigoplus_{n=0}^{\infty} R[x, f]_n$, with $R[x, f]_n = Rx^n$ for all $n \in \mathbb{N}_0$.

If, for $n \in \mathbb{N}$, we endow Rx^n with its natural structure as an (R, R)-bimodule (inherited from its being a graded component of $R[x, f]$), then Rx^n is isomorphic (as (R, R)-bimodule) to R viewed as a left R-module in the natural way and as a right

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R-module via f^n, the nth iterate of the Frobenius ring homomorphism. With this observation, we can formulate the definitions of F-purity and of the tight closure of the zero submodule in an R-module M in terms of the left $R[x,f]$-module $R[x,f] \otimes_R M$, as follows.

First of all, we can write that R is *F-pure* precisely when, for each R-module N, the map $\psi_N : N \longrightarrow Rx \otimes_R N$ for which $\psi_N(g) = x \otimes g$ for all $g \in N$ is injective.

For discussion of tight closure we use R° to denote the complement in R of the union of the minimal prime ideals of R. Let L and M be R-modules and let K be a submodule of L. Observe that there is a natural structure as $\mathbb{N}_0$-graded left $R[x,f]$-module on $R[x,f] \otimes_R M = \bigoplus_{n\in\mathbb{N}_0}(Rx^n \otimes_R M)$. An element $m \in M$ belongs to 0^*_M, the *tight closure of the zero submodule in* M, if and only if there exists $c \in R^\circ$ such that the element $1 \otimes m \in (R[x,f] \otimes_R M)_0$ is annihilated by cx^j for all $j \gg 0$: see M. Hochster and C. Huneke [7, §8]. Furthermore, the tight closure K^*_L of K in L is the inverse image, under the natural epimorphism $L \longrightarrow L/K$, of $0^*_{L/K}$, the tight closure of 0 in L/K.

A test element (for modules) for R is an element $c \in R^\circ$ such that, for every finitely generated R-module M and every $j \in \mathbb{N}_0$, the element cx^j annihilates $1 \otimes m \in (R[x,f] \otimes_R M)_0$ for every $m \in 0^*_M$. If R has a test element, then it must be reduced. It is a result of Hochster and Huneke [8, Theorem (6.1)(b)] that a reduced algebra of finite type over an excellent local ring of characteristic p has a test element.

In attempts to gain a greater understanding of tight closure, the author has investigated in the recent paper [14] properties of certain left modules over the Frobenius skew polynomial ring $R[x,f]$. Let H be a left $R[x,f]$-module. The *graded annihilator* $\operatorname{gr-ann}_{R[x,f]} H$ (or $\operatorname{grann}_{R[x,f]} H$) *of* H is defined in [14, 1.5] and is the largest graded two-sided ideal of $R[x,f]$ that annihilates H. We shall use $\mathcal{G}(H)$ (or $\mathcal{G}_{R[x,f]}(H)$ when it is desirable to emphasize the ring R) to denote the set of all graded annihilators of $R[x,f]$-submodules of H.

When H is x-torsion-free, $\operatorname{gr-ann}_{R[x,f]} H$ has the form $\mathfrak{b}R[x,f] = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{b}x^n$ for some radical ideal $\mathfrak{b}$ of R, and, in that case, we write $\mathcal{I}(H)$ for the set of (necessarily radical) ideals $\mathfrak{c}$ of R for which there is an $R[x,f]$-submodule N of H such that $\operatorname{gr-ann}_{R[x,f]} N = \mathfrak{c}R[x,f]$; in these circumstances, the members of $\mathcal{I}(H)$ are referred to as the *H-special R-ideals*, and we have $\mathcal{G}_{R[x,f]}(H) = \{\mathfrak{b}R[x,f] : \mathfrak{b} \in \mathcal{I}(H)\}$. One of the main results of [14] is Corollary 3.11, which states that $\mathcal{I}(H)$ is finite if H is (x-torsion-free and) Artinian (or Noetherian) as an R-module.

In [14, §4], the author applied these ideas in the case where R is an F-injective Gorenstein local ring, with maximal ideal $\mathfrak{m}$ and dimension d, to the 'top' local cohomology module $H := H^d_{\mathfrak{m}}(R)$ of R. (Recall that every local cohomology module of R has a natural structure as a left $R[x,f]$-module.) The statement that R is F-injective implies that H, with its natural structure as a left $R[x,f]$-module, is x-torsion-free, and this implies that R must be reduced. Note also that, in this case, $H^d_{\mathfrak{m}}(R) \cong E_R(R/\mathfrak{m})$. The ideas of [14, §4] yield the finite set $\mathcal{I}(H)$ of radical ideals of R. Let $\mathfrak{b}$ denote the smallest ideal of positive height in $\mathcal{I}(H)$ (interpret $\operatorname{ht} R$ as ∞). In [14, Corollary 4.7] it was shown that, if c is any element of $\mathfrak{b} \cap R^\circ$, then c is a test element for R, and that $\mathfrak{b}$ is the test ideal $\tau(R)$ of R (that is (in this case), the ideal

of R generated by all test elements of R). It should be noted that these results were obtained without the assumption that R is excellent.

This paper partially generalizes the above-described results of [14, Corollary 4.7] to the case where R is local, with maximal ideal $\mathfrak{m}$, and $E := E_R(R/\mathfrak{m})$ carries a structure of x-torsion-free left $R[x,f]$-module. In this situation, which is more general than that of [14, Corollary 4.7], we again use results from [14, §4] that yield the finite set $\mathcal{I}(E)$ of radical ideals of R. One of the main results of this paper is that, if $\mathfrak{b}$ is the smallest ideal of positive height in $\mathcal{I}(E)$, then each element of $\mathfrak{b} \cap R^\circ$ is a test element (for modules) for R.

A byproduct is an analogue of a result of Janet Cowden Vassilev in [3, §3]: she showed, in the case where R is an F-pure homomorphic image of an F-finite regular local ring, that there exists a strictly ascending chain $0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_t = R$ of radical ideals of R such that, for each $i = 0, \ldots, t-1$, the reduced local ring R/τ_i is F-pure and its test ideal (has positive height and) is exactly τ_{i+1}/τ_i. This paper presents an analogous result in the case where R is complete (but not necessarily F-finite) and E has a structure as an x-torsion-free left $R[x,f]$-module.

1. Graded left modules over the Frobenius skew polynomial ring

This paper builds on the results of [14], and we shall use the notation and terminology of §1 of that paper.

1.1. **Notation.** The notation introduced in the Introduction will be maintained.

The symbols $\mathfrak{a}$ and $\mathfrak{b}$ will always denote ideals of R. We shall only assume that R is local when this is explicitly stated; then, the notation '$(R,\mathfrak{m})$' will denote that $\mathfrak{m}$ is the maximal ideal of R.

Let H be a left $R[x,f]$-module. Recall from [14, 1.5] that an $R[x,f]$-submodule of H is said to be a *special annihilator submodule of* H if it has the form $\operatorname{ann}_H(\mathfrak{B})$ for some *graded* two-sided ideal $\mathfrak{B}$ of $R[x,f]$. As in [14], we shall use $\mathcal{A}(H)$ to denote the set of special annihilator submodules of H.

The *x-torsion submodule* $\Gamma_x(H)$ *of* H, and the concept of *x-torsion-free left* $R[x,f]$*-module,* were defined in [14, 1.2]. The left $R[x,f]$-module $H/\Gamma_x(H)$ is automatically x-torsion-free.

For $n \in \mathbb{Z}$, we shall denote the nth component of a $\mathbb{Z}$-graded left $R[x,f]$-module G by G_n. If $\phi : L \longrightarrow M$ is a homogeneous homomorphism of $\mathbb{Z}$-graded left $R[x,f]$-modules (of degree 0), then the notation $\phi = \bigoplus_{n\in\mathbb{Z}} \phi_n : \bigoplus_{n\in\mathbb{Z}} L_n \longrightarrow \bigoplus_{n\in\mathbb{Z}} M_n$ will indicate that $\phi_n : L_n \longrightarrow M_n$ is the restriction of ϕ to L_n (for all $n \in \mathbb{Z}$). For $t \in \mathbb{Z}$, we shall denote the *tth shift functor* on the category of ($\mathbb{Z}$-)graded left $R[x,f]$-modules and homogeneous homomorphisms (of degree 0) by $(\,\bullet\,)(t)$: thus, for a graded left $R[x,f]$-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$, we have $(M(t))_n = M_{n+t}$ for all $n \in \mathbb{Z}$; also, $f(t)\lceil_{(M(t))_n} = f\lceil_{M_{n+t}}$ for each morphism f in the above-mentioned category and all $n \in \mathbb{Z}$.

In this paper, much use will be made of results in §1 and §3 of [14]. There now follow brief reminders of some of those results that are particularly important for this paper.

1.2. **Lemma** ([14, Lemma 1.7(v)]). *There is an order-reversing bijection, Γ, from the set $\mathcal{A}(H)$ of special annihilator submodules of H to the set $\mathcal{G}(H)$ of graded annihilators of submodules of H given by*

$$\Gamma : N \longmapsto \text{gr-ann}_{R[x,f]} N.$$

The inverse bijection, Γ^{-1}, also order-reversing, is given by

$$\Gamma^{-1} : \mathfrak{B} \longmapsto \text{ann}_H(\mathfrak{B}).$$

1.3. **Lemma** ([14, Lemma 1.9]). *Let H be an x-torsion-free left $R[x,f]$-module. Then there is a radical ideal $\mathfrak{b}$ of R such that $\text{gr-ann}_{R[x,f]} H = \mathfrak{b}R[x,f] = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{b}x^n$.*

1.4. **Proposition** ([14, Proposition 1.11]). *Let H be an x-torsion-free left $R[x,f]$-module.*

There is an order-reversing bijection, $\Delta : \mathcal{A}(H) \longrightarrow \mathcal{I}(H)$, from the set $\mathcal{A}(H)$ of special annihilator submodules of H to the set $\mathcal{I}(H)$ of H-special R-ideals given by

$$\Delta : N \longmapsto \left(\text{gr-ann}_{R[x,f]} N\right) \cap R = (0 :_R N).$$

The inverse bijection, $\Delta^{-1} : \mathcal{I}(H) \longrightarrow \mathcal{A}(H)$, also order-reversing, is given by

$$\Delta^{-1} : \mathfrak{b} \longmapsto \text{ann}_H\left(\mathfrak{b}R[x,f]\right)).$$

When $N \in \mathcal{A}(H)$ and $\mathfrak{b} \in \mathcal{I}(H)$ are such that $\Delta(N) = \mathfrak{b}$, we shall say simply that 'N and $\mathfrak{b}$ correspond'.

1.5. **Corollary** ([14, Corollary 3.7]). *Let H be an x-torsion-free left $R[x,f]$-module. Then the set of H-special R-ideals is precisely the set of all finite intersections of prime H-special R-ideals (provided one includes the empty intersection, R, which corresponds to the zero special annihilator submodule of H). In symbols,*

$$\mathcal{I}(H) = \{\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_t : t \in \mathbb{N}_0 \textit{ and } \mathfrak{p}_1, \ldots, \mathfrak{p}_t \in \mathcal{I}(H) \cap \text{Spec}(R)\}.$$

1.6. **Theorem** ([14, Corollary 3.11]). *Suppose that H is an x-torsion-free left $R[x,f]$-module that is either Artinian or Noetherian as an R-module. Then the set $\mathcal{I}(H)$ of H-special R-ideals is finite.*

By Proposition 1.4, for an x-torsion-free left $R[x,f]$-module H, the set $\mathcal{I}(H)$ is finite if and only if the set $\mathcal{A}(H)$ of special annihilator submodules of H is finite.

The terminology in the following definition is inspired by the Hartshorne–Speiser–Lyubeznik Theorem: see Lyubeznik [12, Proposition 4.4] and compare R. Hartshorne and R. Speiser [6, Proposition 1.11].

1.7. **Definition** ([14, Definition 3.15]). Let H be a left $R[x,f]$-module. We say that H *admits an HSL-number* if there exists $e \in \mathbb{N}_0$ such that $x^e\Gamma_x(H) = 0$; then we call the smallest such e the *HSL-number* of H, and denote this by $\text{HSL}(H)$.

If there is no such non-negative integer, that is, if H does not admit an HSL-number, then we shall write $\text{HSL}(H) = \infty$.

In this terminology, the conclusion of the Hartshorne–Speiser–Lyubeznik Theorem is that, when $(R,\mathfrak{m})$ is local, a left $R[x,f]$-module that is Artinian as an R-module admits an HSL-number.

We end this section by showing that the results from [14] quoted above lead quickly to a generalization of a result of F. Enescu and M. Hochster [4, Theorem 3.7].

1.8. **Proposition.** *Suppose that the local ring $(R, \mathfrak{m})$ is complete, and suppose that $E := E_R(R/\mathfrak{m})$ has a structure as left $R[x, f]$-module. Then every $R[x, f]$-submodule of E is a special annihilator submodule.*

Consequently, if E is x-torsion-free, then there are only finitely many $R[x, f]$-submodules of E, and, for each E-special R-ideal $\mathfrak{b}$, we have $\operatorname{ann}_E \left(\bigoplus_{n\in\mathbb{N}_0} \mathfrak{b}x^n\right) = \operatorname{ann}_E(\mathfrak{b})$.

Proof. Let L be an $R[x, f]$-submodule of E, and let $\mathfrak{B}$ be its graded annihilator. Thus there is an increasing sequence $(\mathfrak{b}_n)_{n\in\mathbb{N}_0}$ of ideals of R such that $\mathfrak{B} = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{b}_n x^n$, and $\mathfrak{b}_0 = (0 :_R L)$. Now, by Matlis duality (see, for example, [15, p. 154]), every R-submodule M of E satisfies $M = \operatorname{ann}_E((0 :_R M))$. Therefore

$$\begin{aligned} L &\subseteq \operatorname{ann}_E(\operatorname{gr-ann}_{R[x,f]} L) \\ &= \operatorname{ann}_E \left(\textstyle\bigoplus_{n\in\mathbb{N}_0} \mathfrak{b}_n x^n\right) \subseteq \operatorname{ann}_E(\mathfrak{b}_0) = \operatorname{ann}_E((0 :_R L)) = L. \end{aligned}$$

Therefore $L = \operatorname{ann}_E(\operatorname{gr-ann}_{R[x,f]} L) = \operatorname{ann}_E \left(\bigoplus_{n\in\mathbb{N}_0} \mathfrak{b}_n x^n\right) = \operatorname{ann}_E(\mathfrak{b}_0)$ and L is a special annihilator submodule of E.

For the claims in the final paragraph, note that it follows from the above that

$$\operatorname{ann}_E \left(\textstyle\bigoplus_{n\in\mathbb{N}_0} \mathfrak{b}x^n\right) = \operatorname{ann}_E(\mathfrak{b}),$$

and, by [14, Corollary 3.11] and the fact that E is Artinian as an R-module, there are only finitely many special annihilator submodules of E. □

1.9. **Corollary.** *Suppose that $(R, \mathfrak{m})$ is local, and that $E := E_R(R/\mathfrak{m})$ has a structure as an x-torsion-free left $R[x, f]$-module. Then there are only finitely many $R[x, f]$-submodules of E.*

Proof. Denote the completion of R by $(\widehat{R}, \widehat{\mathfrak{m}})$. Recall the natural $\widehat{R}$-module structure on the Artinian R-module E: given $h \in E$, there exists $t \in \mathbb{N}$ such that $\mathfrak{m}^t h = 0$; for an $\widehat{r} \in \widehat{R}$, choose any $r \in R$ such that $\widehat{r} - r \in \mathfrak{m}^t\widehat{R}$; then $\widehat{r}h = rh$. It is easy to see from this that $x\widehat{r}h = \widehat{r}^p xh$ for all $h \in E$ and $\widehat{r} \in \widehat{R}$; thus E inherits a structure as left $\widehat{R}[x, f]$-module that extends both its $R[x, f]$-module and $\widehat{R}$-module structures. In particular, E is x-torsion-free as left $\widehat{R}[x, f]$-module.

It is easy to use [2, 10.2.10] (for example) to see that, when E is regarded as an $\widehat{R}$-module in this way, then it is isomorphic to $E_{\widehat{R}}(\widehat{R}/\widehat{\mathfrak{m}})$. Since a subset of E is an $R[x, f]$-submodule if and only if it is an $\widehat{R}[x, f]$-submodule, it is enough for us to prove the claim under the additional assumption that R is complete; in that case, the desired conclusion follows from Proposition 1.8. □

As a corollary, we obtain a result that has already been established by F. Enescu and M. Hochster. Recall that the d-dimensional local ring $(R, \mathfrak{m})$ is said to be *quasi-Gorenstein* precisely when the top local cohomology module $H^d_{\mathfrak{m}}(R)$ is isomorphic to $E_R(R/\mathfrak{m})$.

1.10. **Corollary** (Enescu–Hochster [4, Theorem 3.7]). *Suppose that the d-dimensional local ring $(R, \mathfrak{m})$ is quasi-Gorenstein and F-pure. Then $H^d_{\mathfrak{m}}(R)$, regarded as an $R[x, f]$-module in the natural way, has only finitely many $R[x, f]$-submodules.*

In particular, if $(R, \mathfrak{m})$ is Gorenstein and F-pure, then $H^d_{\mathfrak{m}}(R)$ has only finitely many $R[x, f]$-submodules.

Proof. This follows from Corollary 1.9 because $H^d_{\mathfrak{m}}(R) \cong E_R(R/\mathfrak{m})$ and the hypothesis that R is F-pure ensures that the natural $R[x,f]$-module structure on $H^d_{\mathfrak{m}}(R)$ is x-torsion-free. □

2. New left $R[x,f]$-modules from old

The purpose of this section is to introduce some methods for the construction of new left $R[x,f]$-modules from old, and to describe the sets of graded annihilators, and the HSL-numbers, of (some of) these new modules in terms of the corresponding invariants of the original modules. Although the main applications of these ideas in this paper will be to left $R[x,f]$-modules which are x-torsion-free, it is convenient to note at the same time conclusions that apply in the more general case.

2.1. **Lemma.** *Let $\left(H^{(\lambda)}\right)_{\lambda\in\Lambda}$ be a non-empty family of $\mathbb{Z}$-graded left $R[x,f]$-modules, with gradings given by $H^{(\lambda)} = \bigoplus_{n\in\mathbb{Z}} H^{(\lambda)}_n$ for each $\lambda \in \Lambda$. For each $n \in \mathbb{Z}$, set $H_n := \prod_{\lambda\in\Lambda} H^{(\lambda)}_n$. Then the R-module*

$$H := \bigoplus_{n\in\mathbb{Z}} H_n = \bigoplus_{n\in\mathbb{Z}} \left(\prod_{\lambda\in\Lambda} H^{(\lambda)}_n \right)$$

has a natural structure as a ($\mathbb{Z}$-graded) left $R[x,f]$-module in which

$$x\left(h^{(\lambda)}_n\right)_{\lambda\in\Lambda} = \left(xh^{(\lambda)}_n\right)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} H^{(\lambda)}_{n+1} \quad \textit{for all } \left(h^{(\lambda)}_n\right)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} H^{(\lambda)}_n.$$

This graded left $R[x,f]$-module H is, in fact, the product of $\left(H^{(\lambda)}\right)_{\lambda\in\Lambda}$ in the category of $\mathbb{Z}$-graded left $R[x,f]$-modules and homogeneous $R[x,f]$-homomorphisms (of degree 0); however, to avoid possible confusion, we shall denote the module H by $\prod'_{\lambda\in\Lambda} H^{(\lambda)}$.

Proof. This is straightforward and will be left to the reader; one can use [11, Lemma 1.3] to facilitate the verification that H has a structure as a left $R[x,f]$-module as claimed. □

2.2. *Remark.* Let $\left(H^{(\lambda)}\right)_{\lambda\in\Lambda}$ be a non-empty family of $\mathbb{Z}$-graded left $R[x,f]$-modules, and, as in Lemma 2.1, set $H := \prod'_{\lambda\in\Lambda} H^{(\lambda)}$. Let $\mathfrak{B}$ be a graded two-sided ideal of $R[x,f]$. Then it is straightforward to check that

(i) $\operatorname{ann}_H \mathfrak{B} = \prod'_{\lambda\in\Lambda} \operatorname{ann}_{H^{(\lambda)}} \mathfrak{B}$, and
(ii) $\operatorname{gr-ann}_{R[x,f]} H = \bigcap_{\lambda\in\Lambda} \operatorname{gr-ann}_{R[x,f]} H^{(\lambda)}$.

2.3. **Lemma.** *Let $\left(H^{(\lambda)}\right)_{\lambda\in\Lambda}$ be a non-empty family of $\mathbb{Z}$-graded left $R[x,f]$-modules, and, as in Lemma 2.1, set $H := \prod'_{\lambda\in\Lambda} H^{(\lambda)}$.*

(i) *We have $\mathcal{G}(H) = \left\{\bigcap_{\lambda\in\Lambda} \mathfrak{B}_\lambda : \mathfrak{B}_\lambda \in \mathcal{G}(H^{(\lambda)}) \text{ for all } \lambda \in \Lambda\right\}$.*
(ii) *Consequently, if there exists a set $\mathcal{G}'$ of graded two-sided ideals of $R[x,f]$ such that $\mathcal{G}(H^{(\lambda)}) = \mathcal{G}'$ for all $\lambda \in \Lambda$, then $\mathcal{G}(H) = \mathcal{G}'$.*
(iii) *We have $\operatorname{HSL}(H) = \sup\left\{\operatorname{HSL}(H^{(\lambda)}) : \lambda \in \Lambda\right\}$ (even if $\operatorname{HSL}(H^{(\mu)}) = \infty$ for some $\mu \in \Lambda$). Thus H is x-torsion-free if and only if $H^{(\lambda)}$ is x-torsion-free for all $\lambda \in \Lambda$.*

(iv) *If* $\mathrm{HSL}(H)$ *is finite, then* $\Gamma_x(H) = \prod'_{\lambda\in\Lambda} \Gamma_x(H^{(\lambda)})$ *and there is a homogeneous isomorphism of graded left* $R[x,f]$*-modules*

$$H/\Gamma_x(H) \xrightarrow{\cong} \prod_{\lambda\in\Lambda}{}' H^{(\lambda)}/\Gamma_x(H^{(\lambda)}).$$

Proof. (i) Let $\mathfrak{B} \in \mathcal{G}(H)$. Then, by 1.2 and 2.2, we have

$$\begin{aligned}\mathfrak{B} &= \text{gr-ann}_{R[x,f]}(\mathrm{ann}_H(\mathfrak{B})) = \text{gr-ann}_{R[x,f]}\left(\prod_{\lambda\in\Lambda}{}' \mathrm{ann}_{H^{(\lambda)}} \mathfrak{B}\right)\\ &= \bigcap_{\lambda\in\Lambda} \text{gr-ann}_{R[x,f]}(\mathrm{ann}_{H^{(\lambda)}} \mathfrak{B})\\ &\in \left\{\bigcap_{\lambda\in\Lambda} \mathfrak{B}_\lambda : \mathfrak{B}_\lambda \in \mathcal{G}(H^{(\lambda)}) \text{ for all } \lambda \in \Lambda\right\}.\end{aligned}$$

Next, for each $\lambda \in \Lambda$, let $\mathfrak{B}_\lambda \in \mathcal{G}(H^{(\lambda)})$. Then $\mathfrak{B}_\lambda = \text{gr-ann}_{R[x,f]}(\mathrm{ann}_{H^{(\lambda)}}(\mathfrak{B}_\lambda))$, by 1.2. Set $L := \prod'_{\lambda\in\Lambda} \mathrm{ann}_{H^{(\lambda)}}(\mathfrak{B}_\lambda)$, a graded $R[x,f]$-submodule of H. Then, by 2.2,

$$\text{gr-ann}_{R[x,f]} L = \bigcap_{\lambda\in\Lambda} \text{gr-ann}_{R[x,f]}(\mathrm{ann}_{H^{(\lambda)}}(\mathfrak{B}_\lambda)) = \bigcap_{\lambda\in\Lambda} \mathfrak{B}_\lambda,$$

and so $\bigcap_{\lambda\in\Lambda} \mathfrak{B}_\lambda \in \mathcal{G}(H)$.

(ii) This is now immediate from part (i), because each $\mathcal{G}(H^{(\lambda)})$ (for a $\lambda \in \Lambda$) is closed under taking arbitrary intersections.

(iii),(iv) These are straightforward, and left to the reader. $\square$

2.4. **Lemma.** *Let* H *be a left* $R[x,f]$*-module. Let* $\mathfrak{B}$ *be a two-sided ideal of* $R[x,f]$*. For all* $n \in \mathbb{N}_0$*, set* $H_n := H$*. Then the* R*-module* $\widetilde{H} := \bigoplus_{n\in\mathbb{N}_0} H_n$ *has a natural structure as a graded left* $R[x,f]$*-module under which the result of multiplying* $h_n \in H_n = H$ *on the left by* x *is the element* $xh_n \in H_{n+1} = H$*.*

Furthermore,

(i) $\text{gr-ann}_{R[x,f]}(\widetilde{H}) = \text{gr-ann}_{R[x,f]}(H)$*, and*

(ii) $\mathrm{ann}_{\widetilde{H}} \mathfrak{B} = \bigoplus_{n\in\mathbb{N}_0} \mathrm{ann}_{H_n} \mathfrak{B} = \mathrm{ann}_H \mathfrak{B} \oplus \mathrm{ann}_H \mathfrak{B} \oplus \cdots \oplus \mathrm{ann}_H \mathfrak{B} \oplus \cdots = \widetilde{\mathrm{ann}_H \mathfrak{B}}$.

Proof. One can use [11, Lemma 1.3] to facilitate the verification that $\widetilde{H}$ has a structure as a left $R[x,f]$-module as claimed. The claims in (i) and (ii) are clear. $\square$

2.5. **Lemma.** *Let* H *be a left* $R[x,f]$*-module; set* $G := H/\Gamma_x(H)$*. Let* $\widetilde{H}$ *be the graded left* $R[x,f]$*-module constructed from* H *as in Lemma 2.4. Then*

(i) $\mathcal{G}(\widetilde{H}) = \mathcal{G}(H)$*;*

(ii) $\mathrm{HSL}(\widetilde{H}) = \mathrm{HSL}(H)$ *and* $\Gamma_x(\widetilde{H}) = \bigoplus_{n\in\mathbb{N}_0} \Gamma_x(H_n) = \widetilde{\Gamma_x(H)}$*; and*

(iii) *there is an* $R[x,f]$*-isomorphism*

$$\widetilde{H}/\Gamma_x(\widetilde{H}) \cong \bigoplus_{n\in\mathbb{N}_0} H_n/\Gamma_x(H_n) = \widetilde{G}.$$

Proof. (i) Let $\mathfrak{B} \in \mathcal{G}(\widetilde{H})$. Then, by Lemmas 1.2 and 2.4, we have

$$\mathfrak{B} = \operatorname{gr-ann}_{R[x,f]}(\operatorname{ann}_{\widetilde{H}} \mathfrak{B}) = \operatorname{gr-ann}_{R[x,f]}(\widetilde{\operatorname{ann}_H \mathfrak{B}}) = \operatorname{gr-ann}_{R[x,f]}(\operatorname{ann}_H \mathfrak{B}) \in \mathcal{G}(H).$$

On the other hand, if $\mathfrak{B} \in \mathcal{G}(H)$, then, again by Lemmas 1.2 and 2.4,

$$\mathfrak{B} = \operatorname{gr-ann}_{R[x,f]}(\operatorname{ann}_H \mathfrak{B}) = \operatorname{gr-ann}_{R[x,f]}(\widetilde{\operatorname{ann}_H \mathfrak{B}}) = \operatorname{gr-ann}_{R[x,f]}(\operatorname{ann}_{\widetilde{H}} \mathfrak{B}) \in \mathcal{G}(\widetilde{H}).$$

(ii),(iii) These are straightforward and left to the reader. □

2.6. *Remark.* Suppose that $(R, \mathfrak{m})$ is local. Let $H := H^d_{\mathfrak{m}}(R)$, considered as a left $R[x, f]$-module in the natural way (recalled in [14, Reminder 4.1(ii)]). The isomorphisms described in [14, Remark 4.2(iii)] show that, if we apply the construction of Lemma 2.4 to this H, then the resulting graded left $R[x, f]$-module $\widetilde{H}$ is isomorphic to $R[x, f] \otimes_R H^d_{\mathfrak{m}}(R)$.

2.7. **Notation.** For $t \in \mathbb{N}$, we refer to the mapping $f : R^t \longrightarrow R^t$ for which $f((r_1, \ldots, r_t)) = (r_1^p, \ldots, r_t^p)$ for all $(r_1, \ldots, r_t) \in R^t$ as the *Frobenius map*.

2.8. **Lemma.** *Let $b \in \mathbb{N}$ and $W = \bigoplus_{n \geq b} W_n$ be a $\mathbb{Z}$-graded left $R[x, f]$-module; let $g_1, \ldots, g_t \in W_b$, and let*

$$K := \left\{(r_1, \ldots, r_t) \in R^t : \textstyle\sum_{i=1}^t r_i g_i = 0\right\}.$$

Then there is a graded left $R[x, f]$-module

$$W' = \bigoplus_{n \geq b-1} W'_n = \left(R^t/f^{-1}(K)\right) \oplus W_b \oplus W_{b+1} \oplus \cdots \oplus W_i \oplus \cdots$$

(so that $W'_{b-1} = R^t/f^{-1}(K)$ and $W'_n = W_n$ for all $n \geq b$) which has W as an $R[x, f]$-submodule and for which $x((r_1, \ldots, r_t) + f^{-1}(K)) = \sum_{i=1}^t r_i^p g_i$ for all $(r_1, \ldots, r_t) \in R^t$. We call W' the 1-place extension of W by $g_1, \ldots, g_t$, *and denote it by*

$$\operatorname{exten}(W; g_1, \ldots, g_t; 1).$$

If W is x-torsion-free, then so too is $\operatorname{exten}(W; g_1, \ldots, g_t; 1)$, *and then*

$$\mathcal{G}(\operatorname{exten}(W; g_1, \ldots, g_t; 1)) = \mathcal{G}(W)$$

and $\mathcal{I}(\operatorname{exten}(W; g_1, \ldots, g_t; 1)) = \mathcal{I}(W)$.

Proof. One can use [11, Lemma 1.3] to facilitate the verification that W' is a graded left $R[x, f]$-module.

Suppose that W is x-torsion-free, and let $(r_1, \ldots, r_t) \in R^t$ be such that the element $\eta := (r_1, \ldots, r_t) + f^{-1}(K)$ of W'_{b-1} belongs to $\Gamma_x(W')$. Then there exists $h \in \mathbb{N}$ such that $x^h \eta = x^{h-1}(\sum_{i=1}^t r_i^p g_i) = 0$. Since W is x-torsion-free, we have $\sum_{i=1}^t r_i^p g_i = x\eta = 0$. Therefore $f((r_1, \ldots, r_t)) \in K$, and so $\eta := (r_1, \ldots, r_t) + f^{-1}(K) = 0$. It follows that, if W is x-torsion-free, then W' is x-torsion-free. The converse is clear.

In order to prove the final two claims, it is sufficient to prove that $\mathcal{I}(W') = \mathcal{I}(W)$. Since W is an $R[x, f]$-submodule of W', it is clear that $\mathcal{I}(W) \subseteq \mathcal{I}(W')$. Let $\mathfrak{b} \in \mathcal{I}(W')$, so that $\mathfrak{b}R[x, f]$ is the graded annihilator of the graded $R[x, f]$ submodule $L' := \bigoplus_{n \geq b-1} L'_n = \operatorname{ann}_{W'}(\mathfrak{b}R[x, f])$ of W'. Set $L := \bigoplus_{n \geq b} L'_n$, and let $\operatorname{gr-ann}_{R[x,f]} L = \mathfrak{c}R[x, f]$, where $\mathfrak{c}$ is a radical ideal of R. Note that $\mathfrak{c} \in \mathcal{I}(W)$ and $\mathfrak{b} \subseteq \mathfrak{c}$. Now the two-sided ideal $\bigoplus_{n \geq 1} \mathfrak{c}x^n$ of $R[x, f]$ annihilates L', so that $xcm = c^p xm = 0$ for all

$c \in \mathfrak{c}$ and $m \in L'$. Since L' is x-torsion-free by the above, it follows that $\mathfrak{c}L' = 0$ and $\bigoplus_{n\geq 0} \mathfrak{c}x^n = \mathfrak{c}R[x,f]$ annihilates L'. Therefore $\mathfrak{c} \subseteq \mathfrak{b}$, and $\mathfrak{b} = \mathfrak{c} \in \mathcal{I}(W)$. □

2.9. *Remark.* Here we use the notation and terminology of Lemma 2.8, and write $\overline{W}$ for $W/\Gamma_x(W)$, and also use 'overlines' to denote natural images in $\overline{W}$ of elements of W.

It is straightforward to check that there is a homogeneous isomorphism of graded left $R[x,f]$-modules

$$\operatorname{exten}(W;g_1,\ldots,g_t;1)/\Gamma_x(\operatorname{exten}(W;g_1,\ldots,g_t;1)) \cong \operatorname{exten}(\overline{W};\overline{g_1},\ldots,\overline{g_t};1),$$

so that, by Lemma 2.8,

$$\mathcal{G}\big(\operatorname{exten}(W;g_1,\ldots,g_t;1)/\Gamma_x(\operatorname{exten}(W;g_1,\ldots,g_t;1))\big) = \mathcal{G}(\overline{W})$$

and

$$\mathcal{I}\big(\operatorname{exten}(W;g_1,\ldots,g_t;1)/\Gamma_x(\operatorname{exten}(W;g_1,\ldots,g_t;1))\big) = \mathcal{I}(\overline{W}).$$

2.10. **Definition.** Let $b \in \mathbb{N}$ and $W = \bigoplus_{n\geq b} W_n$ be a $\mathbb{Z}$-graded left $R[x,f]$-module; let $g_1,\ldots,g_t \in W_b$. The 1-place extension $\operatorname{exten}(W;g_1,\ldots,g_t;1)$ of W by $g_1,\ldots,g_t$ was defined in Lemma 2.8. Recall that we defined

$$K := \left\{(r_1,\ldots,r_t) \in R^t : \textstyle\sum_{i=1}^t r_i g_i = 0\right\}.$$

Now let $h \in \mathbb{N}$ with $h \geq 2$. The *h-place extension* $\operatorname{exten}(W;g_1,\ldots,g_t;h)$ *of* W *by* $g_1,\ldots,g_t$ is the graded left $R[x,f]$-module

$$\big(R^t/f^{-h}(K)\big) \oplus \cdots \oplus \big(R^t/f^{-1}(K)\big) \oplus W_b \oplus \cdots \oplus W_i \oplus \cdots$$

which has $\operatorname{exten}(W;g_1,\ldots,g_t;1)$ as a graded $R[x,f]$-submodule and is such that

$$x(v + f^{-j}(K)) = f(v) + f^{-(j-1)}(K) \quad \text{for all } v \in R^t \text{ and } j = h, h-1, \ldots, 2.$$

For each $i \in \{1,\ldots,t\}$, let e_i denote the element $(0,\ldots,0,1,0,\ldots,0)$ of R^t which has a 1 in the ith spot and all other components 0. It is straightforward to check that

$$\operatorname{exten}(W;g_1,\ldots,g_t;h) = \operatorname{exten}(\operatorname{exten}(W;g_1,\ldots,g_t;1);\overline{e_1},\ldots,\overline{e_t};h-1),$$

where, for $v \in R^t$, we use $\overline{v}$ to denote $v + f^{-1}(K)$, and

$$\operatorname{exten}(W;g_1,\ldots,g_t;h) = \operatorname{exten}(\operatorname{exten}(W;g_1,\ldots,g_t;h-1);\widetilde{e_1},\ldots,\widetilde{e_t};1),$$

where, for $v \in R^t$, we use $\widetilde{v}$ to denote $v + f^{-(h-1)}(K)$.

It is a consequence of Lemma 2.8 that, if W is x-torsion-free, then so too is $\operatorname{exten}(W;g_1,\ldots,g_t;h)$, and then $\mathcal{G}(\operatorname{exten}(W;g_1,\ldots,g_t;h)) = \mathcal{G}(W)$ and

$$\mathcal{I}(\operatorname{exten}(W;g_1,\ldots,g_t;h)) = \mathcal{I}(W).$$

2.11. **Proposition.** *Let $b \in \mathbb{N}$ and $W = \bigoplus_{n\geq b} W_n$ be a $\mathbb{Z}$-graded left $R[x,f]$-module, and let M be an R-module generated by the finite set $\{m_1,\ldots,m_t\}$; then we can form the graded $R[x,f]$-submodule $\bigoplus_{i\geq b}(Rx^i \otimes_R M)$ of $R[x,f]\otimes_R M$. Suppose that there is given a homogeneous $R[x,f]$-homomorphism $\lambda' = \bigoplus_{i\geq b}\lambda_i : \bigoplus_{i\geq b}(Rx^i\otimes_R M) \longrightarrow W$.*

For each $j = 1,\ldots,t$, let $g_j := \lambda_b(x^b \otimes m_j) \in W_b$. Set

$$K := \left\{(r_1,\ldots,r_t) \in R^t : \textstyle\sum_{j=1}^t r_j g_j = 0\right\},$$

as in Lemma 2.8. For each $i = 0, 1, \ldots, b-1$, there exists an R-homomorphism $\lambda_i : Rx^i \otimes_R M \longrightarrow R^t/f^{-(b-i)}(K)$ such that

$$\lambda_i \left(\textstyle\sum_{j=1}^{t} r_j x^i \otimes m_j \right) = (r_1, \ldots, r_t) + f^{-(b-i)}(K) \quad \textit{for all } r_1, \ldots, r_t \in R.$$

Furthermore,

$$\lambda := \bigoplus_{i \in \mathbb{N}_0} \lambda_i : R[x,f] \otimes_R M = \bigoplus_{i \in \mathbb{N}_0} (Rx^i \otimes_R M) \longrightarrow \mathrm{exten}(W; g_1, \ldots, g_t; b)$$

is a homogeneous $R[x,f]$-homomorphism that extends λ'.

Proof. This is straightforward once it is been noted that, if $i \in \{0, \ldots, b-1\}$ and $v = (r_1, \ldots, r_t), w = (s_1, \ldots, s_t) \in R^t$ are such that $\sum_{j=1}^{t} r_j x^i \otimes m_j = \sum_{j=1}^{t} s_j x^i \otimes m_j$, then $\sum_{j=1}^{t} r_j^{p^{b-i}} x^b \otimes m_j = \sum_{j=1}^{t} s_j^{p^{b-i}} x^b \otimes m_j$, so that $v - w \in f^{-(b-i)}(K)$ because

$$\begin{aligned} \textstyle\sum_{j=1}^{t} (r_j - s_j)^{p^{b-i}} g_j &= \textstyle\sum_{j=1}^{t} (r_j - s_j)^{p^{b-i}} \lambda_b (x^b \otimes m_j) \\ &= \textstyle\sum_{j=1}^{t} \lambda_b \left((r_j - s_j)^{p^{b-i}} x^b \otimes m_j \right) = 0. \end{aligned}$$

□

3. Use of an $R[x,f]$-module structure on the injective envelope of the simple module over a local ring

3.1. **Lemma.** *Suppose that $(R, \mathfrak{m})$ is local and that there exists a left $R[x,f]$-module E which, as R-module, is isomorphic to $E_R(R/\mathfrak{m})$, the injective envelope of the simple R-module $R/\mathfrak{m}$. Construct the graded left $R[x,f]$-module $\widetilde{E}$ from E, as in Lemma 2.4.*

Let M be a non-zero R-module of finite length with the property that its zero submodule is irreducible, that is, cannot be expressed as the intersection of two non-zero submodules. Then there exists a homogeneous $R[x,f]$-homomorphism

$$\lambda := \bigoplus_{i \in \mathbb{N}_0} \lambda_i : R[x,f] \otimes_R M = \bigoplus_{i \in \mathbb{N}_0} (Rx^i \otimes_R M) \longrightarrow \widetilde{E}$$

such that λ_0 is a monomorphism.

Proof. Since $E_R(M) \cong E_R(R/\mathfrak{m})$, there exists an R-monomorphism $\lambda_0 : M \longrightarrow E$. We can then define, for each $n \in \mathbb{N}$, an R-homomorphism $\lambda_n : Rx^n \otimes_R M \longrightarrow (\widetilde{E})_n = E$ for which $\lambda_n(rx^n \otimes m) = rx^n \lambda_0(m)$ for all $r \in R$ and all $m \in M$. It is straightforward to check that the λ_n ($n \in \mathbb{N}_0$) provide a homogeneous $R[x,f]$-homomorphism as claimed. □

3.2. *Remark.* Let M be an R-module and let $h, n \in \mathbb{N}_0$. Endow Rx^n and Rx^h with their natural structures as (R,R)-bimodules (inherited from their being graded components of $R[x,f]$). Then there is an isomorphism of (left) R-modules $\phi : Rx^{n+h} \otimes_R M \xrightarrow{\cong} Rx^n \otimes_R (Rx^h \otimes_R M)$ for which $\phi(rx^{n+h} \otimes m) = rx^n \otimes (x^h \otimes m)$ for all $r \in R$ and $m \in M$.

3.3. *Remark.* It was pointed out in the Introduction that R is F-pure precisely when, for each R-module N, the map $\psi_N : N \longrightarrow Rx \otimes_R N$ for which $\psi_N(g) = x \otimes g$ for all $g \in N$ is injective. In view of isomorphisms like those described in Remark 3.2 above, this is the case if and only if, for each R-module N, the left $R[x, f]$-module $R[x, f] \otimes_R N$ is x-torsion-free; since tensor product commutes with direct limits, we can conclude that R is F-pure if and only if, for each finitely generated R-module N, the left $R[x, f]$-module $R[x, f] \otimes_R N$ is x-torsion-free.

3.4. **Lemma.** *Let $\mathfrak{d}$ be an ideal of R of positive height. Then $\mathfrak{d}$ can be generated by the elements in $\mathfrak{d} \cap R^{\circ}$.*

Proof. Let $\mathfrak{d}'$ be the ideal generated by the elements of $\mathfrak{d} \cap R^{\circ}$; of course, $\mathfrak{d}' \subseteq \mathfrak{d}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal prime ideals of R. Then $\mathfrak{d} \subseteq \mathfrak{d}' \cup \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t$, and so, by prime avoidance, either $\mathfrak{d} \subseteq \mathfrak{p}_i$ for some $i \in \{1, \ldots, t\}$, or $\mathfrak{d} \subseteq \mathfrak{d}'$. Since $\operatorname{ht} \mathfrak{d} \geq 1$, we must have $\mathfrak{d} \subseteq \mathfrak{d}'$. $\square$

3.5. **Theorem.** *Suppose that $(R, \mathfrak{m})$ is local, and that there exists a left $R[x, f]$-module E which, as R-module, is isomorphic to $E_R(R/\mathfrak{m})$. Let M be a finitely generated R-module. Then, for each $n \in \mathbb{N}_0$, there is a countable family $(E_{ni})_{i \in Y_n}$ of $\mathbb{N}_0$-graded left $R[x, f]$-modules, each x-torsion-free if E is, and with $\mathcal{G}(E_{ni}/\Gamma_x(E_{ni})) = \mathcal{G}(E/\Gamma_x(E))$ for all $i \in Y_n$, for which there exists a homogeneous $R[x, f]$-monomorphism*

$$\nu : R[x, f] \otimes_R M = \bigoplus_{i \in \mathbb{N}_0} (Rx^i \otimes_R M) \longrightarrow {\prod_{\substack{n \in \mathbb{N}_0 \\ i \in Y_n}}}' E_{ni}.$$

In particular, if E is x-torsion-free, then

(i) *R is reduced and has a test element (for modules),*
(ii) *$R[x, f] \otimes_R M$ is x-torsion-free and $\mathcal{I}(R[x, f] \otimes_R M) \subseteq \mathcal{I}(E)$,*
(iii) *R is F-pure,*
(iv) *the unique smallest ideal $\mathfrak{b}$ of positive height in $\mathcal{I}(E)$ is contained in the test ideal $\tau(R)$ of R, and*
(v) *$\tau(R) \in \mathcal{I}(E)$.*

Proof. For each $n \in \mathbb{N}_0$, the (left) R-module $Rx^n \otimes_R M$ is finitely generated, and so $\bigcap_{j \in \mathbb{N}} \mathfrak{m}^j (Rx^n \otimes_R M) = 0$; therefore the zero submodule of $Rx^n \otimes_R M$ can be expressed as the intersection of a countable family $(Q_{ni})_{i \in Y_n}$ of irreducible submodules of finite colength.

Construct the graded left $R[x, f]$-module $\widetilde{E}$ from E, as in Lemma 2.4. By Lemma 3.1, there is, for each $n \in \mathbb{N}_0$ and $i \in Y_n$, a homogeneous $R[x, f]$-homomorphism $R[x, f] \otimes_R ((Rx^n \otimes_R M)/Q_{ni}) \longrightarrow \widetilde{E}$ which is monomorphic in degree 0. If we now compose this with the natural homogeneous $R[x, f]$-epimorphism

$$R[x, f] \otimes_R (Rx^n \otimes_R M) \longrightarrow R[x, f] \otimes_R ((Rx^n \otimes_R M)/Q_{ni})$$

and use isomorphisms of the type described in Remark 3.2, we obtain (after application of the shift functor $(\,\bullet\,)(-n)$) a homogeneous $R[x, f]$-homomorphism

$$\lambda'_{ni} : \bigoplus_{j \geq n} (Rx^j \otimes_R M) \longrightarrow \widetilde{E}(-n)$$

for which the nth component has kernel equal to Q_{ni}.

We can now use Corollary 2.11 to extend λ'_{ni} by n places to produce a homogeneous $R[x,f]$-homomorphism

$$\lambda_{ni} : \bigoplus_{j \geq 0} (Rx^j \otimes_R M) = R[x,f] \otimes_R M \longrightarrow E_{ni},$$

where E_{ni} is an appropriate n-place extension of $\widetilde{E}(-n)$, for which the nth component has kernel equal to Q_{ni}. Note that, by Lemma 2.5, Remark 2.9 and Definition 2.10, we have $\mathcal{G}(E_{ni}/\Gamma_x(E_{ni})) = \mathcal{G}(E/\Gamma_x(E))$ and that E_{ni} is x-torsion-free if E is.

There is therefore a homogeneous $R[x,f]$-homomorphism

$$\nu = \bigoplus_{j \in \mathbb{N}_0} \nu_j : R[x,f] \otimes_R M \longrightarrow \prod_{\substack{n \in \mathbb{N}_0 \\ i \in Y_n}}{}' E_{ni} =: K$$

such that $\nu_j(\xi) = \big((\lambda_{ni})_j(\xi)\big)_{n \in \mathbb{N}_0, i \in Y_n}$ for all $j \in \mathbb{N}_0$ and $\xi \in Rx^j \otimes_R M$. For each $j \in \mathbb{N}_0$, the zero submodule of $Rx^j \otimes_R M$ is equal to $\bigcap_{i \in Y_j} Q_{ji}$, and this means that ν_j is a monomorphism. Hence ν is an $R[x,f]$-monomorphism.

Now suppose for the remainder of the proof that E is x-torsion-free. By Lemmas 2.5 and 2.8, we see that E_{ni} is x-torsion-free, for all $n \in \mathbb{N}_0$ and all $i \in Y_n$. We now deduce from Lemma 2.3(iii) that K is x-torsion-free, so that $R[x,f] \otimes_R M$ is x-torsion-free in view of the $R[x,f]$-monomorphism ν. As this is true for each choice of finitely generated R-module M, it follows from Remark 3.3 that R is F-pure, and therefore reduced.

The $R[x,f]$-monomorphism ν also shows that $\mathcal{G}(R[x,f] \otimes_R M) \subseteq \mathcal{G}(K)$, while Lemmas 2.3(ii), 2.5 and 2.8 show that $\mathcal{G}(K) = \mathcal{G}(\widetilde{E}) = \mathcal{G}(E)$. Therefore

$$\mathcal{I}(R[x,f] \otimes_R M) \subseteq \mathcal{I}(E).$$

Since $E_R(R/\mathfrak{m})$ is Artinian as an R-module, it follows from [14, Corollary 3.11 and Theorem 3.12] that there exists a unique smallest ideal $\mathfrak{b}$ of positive height in $\mathcal{I}(E)$, and that any element of $R[x,f] \otimes_R M$ that is annihilated by $\bigoplus_{n \geq n_0} Rcx^n$ for some $c \in R^\circ$ and $n_0 \in \mathbb{N}_0$ must also be annihilated by $\mathfrak{b}R[x,f]$. This means that each element of $\mathfrak{b} \cap R^\circ$ (note that this set generates $\mathfrak{b}$, by Lemma 3.4) is a test element (for modules) for R.

Next, note that $\operatorname{ann}_{R[x,f]\otimes_R M} \tau(R)R[x,f] = \bigoplus_{n \in \mathbb{N}_0} 0^*_{Rx^n \otimes_R M}$, and so, by the immediately preceding paragraph, the graded annihilator of this will be $\mathfrak{b}_M R[x,f]$ for some $\mathfrak{b}_M \in \mathcal{I}(E)$ for which $\tau(R) \subseteq \mathfrak{b}_M$.

By [14, Corollary 3.7], each member of $\mathcal{I}(E)$ is the intersection of the members of a subset of the finite set $\mathcal{I}(E) \cap \operatorname{Spec}(R)$. It therefore follows that $\tau(R)$ is the intersection of all members of the finite set $\{\mathfrak{p} \in \mathcal{I}(E) \cap \operatorname{Spec}(R) : \operatorname{ht} \mathfrak{p} \geq 1\}$, and that $\mathfrak{b}_M$ is the intersection of the members of some subset of this set.

Set $\mathfrak{d} := \bigcap_M \mathfrak{b}_M$, where the intersection is taken over all finitely generated R-modules M. It follows from the above paragraph that $\mathfrak{d}$ is the intersection of the members of some subset of $\{\mathfrak{p} \in \mathcal{I}(E) \cap \operatorname{Spec}(R) : \operatorname{ht} \mathfrak{p} \geq 1\}$, so that it belongs to $\mathcal{I}(E)$ by [14, Corollary 1.12]. Note that $\operatorname{ht} \mathfrak{d} \geq 1$.

We suppose that $\tau(R) \subset \mathfrak{d}$ and seek a contradiction. (The symbol '$\subset$' is reserved to denote strict inclusion.) Now $\mathfrak{d}$ can, by Lemma 3.4, be generated by elements in $\mathfrak{d} \cap R^\circ$, and so there exists $a \in \mathfrak{d} \cap R^\circ \setminus \tau(R)$. Then a is not a test element for R, and yet it

belongs to R°. This means that there must exist a finitely generated R-module N and an element $y \in 0^*_N$ such that $1 \otimes y \in (R[x,f] \otimes_R N)_0$ is not annihilated by $(aR)R[x,f]$. Therefore $a \notin \mathfrak{b}_N$, and this is a contradiction. Therefore $\tau(R) = \mathfrak{d} \in \mathcal{I}(E)$. □

3.6. *Remark.* In the special case of Theorem 3.5 in which E is x-torsion-free, the main thrust of the argument presented in the above proof comes from the use of the homogeneous $R[x,f]$-monomorphism ν, in conjunction with the unique smallest ideal $\mathfrak{b}$ of positive height in $\mathcal{I}(E)$, to establish the existence of a test element for R. The conclusion (in part (iii)) that R is F-pure comes as a byproduct. I am grateful to the referee for pointing out the following short alternative proof of the fact that, in these circumstances, R is F-pure.

Let $\kappa : E \longrightarrow Rx \otimes_R E$ be the Abelian group homomorphism for which $\kappa(e) = x \otimes e$ for all $e \in E$, and let $\mu : Rx \otimes_R E \longrightarrow E$ be the R-homomorphism for which $\mu(rx \otimes e) = rxe$ for all $r \in R$ and $e \in E$. Then the map $\mu\kappa : E \longrightarrow E$ satisfies $\mu\kappa(e) = xe$ for all $e \in E$, and so is injective because E is x-torsion-free. Hence κ is injective. Therefore R is F-pure by a result of M. Hochster and J. L. Roberts [9, Proposition 6.11].

Before we deduce the main result of the paper from Theorem 3.5, we draw attention to other known results that relate the test ideal of the local ring $(R, \mathfrak{m})$ to annihilators of submodules of the R-module $E(R/\mathfrak{m})$.

3.7. *Discussion.* Recall from Hochster–Huneke [7, Definition (8.22)] that, in general, even in the case where R does not have a test element, the test ideal $\tau(R)$ of R is defined to be $\bigcap_M (0 :_R 0^*_M)$, where the intersection is taken over all finitely generated R-modules M. If R has a test element (for modules), then $\tau(R)$ is the ideal of R generated by all such test elements, and $\tau(R) \cap R^\circ$ is the set of all test elements (for modules) for R. (See [7, Proposition (8.23)(b)].)

Henceforth in this discussion, we assume that $(R, \mathfrak{m})$ is local, and we set $E := E_R(R/\mathfrak{m})$.

(i) Recall from Hochster–Huneke [7, Definition (8.19)] that the *finitistic tight closure of* 0 *in* E, denoted by $0^{*\mathrm{fg}}_E$, is defined to be $\bigcup_M 0^*_M$, where the union is taken over all finitely generated R-submodules M of E. It was shown in [7, Proposition (8.23)(d)] that $\tau(R) = (0 :_R 0^{*\mathrm{fg}}_E)$.

(ii) In the case where R is a reduced ring that is a homomorphic image of an excellent regular local ring of characteristic p, results of G. Lyubeznik and K. E. Smith in [13, §7] establish some very good properties of the ideal

$$\widetilde{\tau}(R) = (0 :_R 0^*_E).$$

Some of their results were extended by I. M. Aberbach and F. Enescu, in [1, Theorem 3.6], to the case where R is a reduced excellent local ring.

Notice that, by part (i), the ideal $\widetilde{\tau}(R)$ is equal to the test ideal of R in the case where $0^*_E = 0^{*\mathrm{fg}}_E$.

(iii) In the case where R is Gorenstein, or merely quasi-Gorenstein, we have $E \cong H^{\dim R}_{\mathfrak{m}}(R)$; then, provided R is excellent and equidimensional (Gorenstein local rings are automatically equidimensional), it follows from work of K. E.

Smith [16, Proposition 3.3] that $0^*_E = 0^{*\mathrm{fg}}_E$, so that, by part (ii),

$$\tau(R) = (0 :_R 0^*_E).$$

(iv) Recall from the Introduction that membership of 0^*_E is (essentially) defined in terms of the natural (graded) left $R[x,f]$-module structure on $R[x,f] \otimes_R E = \bigoplus_{n \in \mathbb{N}_0}(Rx^n \otimes_R E)$: we have that $m \in E$ belongs to 0^*_E if and only if there exists $c \in R^\circ$ and $n_0 \in \mathbb{N}_0$ such that the element

$$1 \otimes m \in (R[x,f] \otimes_R E)_0$$

is annihilated by $\bigoplus_{j \geq n_0} Rcx^j$. In the case where R is Gorenstein, or merely quasi-Gorenstein, we have $E \cong H^{\dim R}_{\mathfrak{m}}(R) =: H$, and the latter R-module carries a natural structure as left $R[x,f]$-module (as recalled in [14, Reminder 4.1]). If we construct the graded left $R[x,f]$-module $\widetilde{H}$ from H as in Lemma 2.4, then it follows from [14, Remark 4.2(iii)] that there are $R[x,f]$-isomorphisms

$$R[x,f] \otimes_R E \cong R[x,f] \otimes_R H \cong \widetilde{H}.$$

(v) Thus, in the special case in which R is Gorenstein, the conclusions of Theorem 3.5(i)–(v) will not surprise experts in tight closure theory; however, in more general situations, the approach taken in that theorem presents a new perspective on the test ideal.

3.8. **Corollary.** *Suppose that $(R,\mathfrak{m})$ is local and complete, and that there exists an x-torsion-free left $R[x,f]$-module E which, as R-module, is isomorphic to $E_R(R/\mathfrak{m})$. Then, for each ideal $\mathfrak{c} \in \mathcal{I}(E)$, the following hold:*

(i) *there is an isomorphism of $R/\mathfrak{c}$-modules $\operatorname{ann}_E(\mathfrak{c}) \cong E_{R/\mathfrak{c}}((R/\mathfrak{c})/(\mathfrak{m}/\mathfrak{c}))$, and $\operatorname{ann}_E(\mathfrak{c})$ inherits from E a structure as x-torsion-free left $(R/\mathfrak{c})[x,f]$-module for which*

$$\mathcal{I}_{R/\mathfrak{c}}(\operatorname{ann}_E(\mathfrak{c})) = \{\mathfrak{d}/\mathfrak{c} : \mathfrak{d} \in \mathcal{I}(E) \text{ and } \mathfrak{d} \supseteq \mathfrak{c}\}\,;$$

(ii) *the complete local ring $\overline{R} := R/\mathfrak{c}$ is F-pure and reduced, and has a test element (for modules); and*

(iii) *the test ideal of $\overline{R}$ is $\mathfrak{d}/\mathfrak{c}$ for some ideal $\mathfrak{d} \in \mathcal{I}(E)$ with $\operatorname{ht}(\mathfrak{d}/\mathfrak{c}) \geq 1$.*

We conclude that there is a strictly ascending chain

$$0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_{n-1} \subset \tau_n = R$$

of ideals of R, all belonging to $\mathcal{I}(E)$, such that, for all $i = 0, \ldots, n-1$, the ring R/τ_i is F-pure and reduced, and has a test element (for modules), and its test ideal is τ_{i+1}/τ_i.

Notes. The referee has pointed out that, in the special case of Corollary 3.8 in which R is (also) Gorenstein and F-finite, the conclusion that $R/\mathfrak{c}$ is F-pure in part (ii) follows from Enescu and Hochster [4, Discussion 2.6 and Theorem 4.1] (and Proposition 1.8 above).

Note that the final conclusion of Corollary 3.8 is similar to J. Cowden Vassilev's result in [3, §4] that, if R is a (not necessarily complete) F-pure homomorphic image

of an F-finite regular local ring, then there exists a strictly ascending chain

$$0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_t = R$$

of radical ideals of R such that, for each $i = 0, \ldots, t-1$, the reduced local ring R/τ_i is F-pure and its test ideal (has positive height and) is exactly τ_{i+1}/τ_i.

Proof. By Proposition 1.8, we have $\operatorname{ann}_E\left(\bigoplus_{n\in\mathbb{N}_0} \mathfrak{c}x^n\right) = \operatorname{ann}_E(\mathfrak{c})$, and this is an $R[x,f]$-submodule of E that is annihilated by $\mathfrak{c}$. Use overlines to denote natural images in $R/\mathfrak{c}$ of elements of R. It is easy to use [11, Lemma 1.3] to see that $\operatorname{ann}_E(\mathfrak{c})$ inherits a structure as left $(R/\mathfrak{c})[x,f]$-module with $\overline{r}e = re$ for all $r \in R$ and $e \in \operatorname{ann}_E(\mathfrak{c})$ (and multiplication by x on an element of $\operatorname{ann}_E(\mathfrak{c})$ as in the $R[x,f]$-module E). Note that $\operatorname{ann}_E(\mathfrak{c})$ is x-torsion-free, and that

$$\mathcal{I}_{R/\mathfrak{c}}(\operatorname{ann}_E(\mathfrak{c})) = \{\mathfrak{d}/\mathfrak{c} : \mathfrak{d} \in \mathcal{I}(E) \text{ and } \mathfrak{d} \supseteq \mathfrak{c}\}.$$

Now, as $R/\mathfrak{c}$-module, $\operatorname{ann}_E(\mathfrak{c}) \cong E_{R/\mathfrak{c}}((R/\mathfrak{c})/(\mathfrak{m}/\mathfrak{c}))$ (by [2, Lemma 10.1.15], for example). Thus part (i) is proved. We can also apply Theorem 3.5 to the left $(R/\mathfrak{c})[x,f]$-module $\operatorname{ann}_E(\mathfrak{c})$ to prove parts (ii) and (iii); the same theorem shows that the test ideal $\tau(R/\mathfrak{c})$ of $R/\mathfrak{c}$ belongs to $\mathcal{I}_{R/\mathfrak{c}}(\operatorname{ann}_E(\mathfrak{c}))$. Note also that $R/\mathfrak{c}$ satisfies the hypotheses of the corollary, so that conclusions (i), (ii) and (iii) are valid for $R/\mathfrak{c}$ and $\operatorname{ann}_E(\mathfrak{c})$; with these observations, it is easy to prove the final claim by induction. □

We end the paper with some comments about sources of examples of local rings that satisfy the hypotheses of Corollary 3.8.

3.9. **Example.** Suppose that $(R,\mathfrak{m})$ is local, complete, F-injective and quasi-Gorenstein. (Note that Gorenstein local rings are quasi-Gorenstein, and that the properties of being F-injective and quasi-Gorenstein are inherited by the completion of a non-complete local ring of characteristic p.)

Let $d := \dim R$ and $H := H^d_{\mathfrak{m}}(R)$, and note that H has a natural left $R[x,f]$-module structure (recalled in [14, Reminder 4.1]); since R is F-injective, H is x-torsion-free; and since R is quasi-Gorenstein, there is an R-isomorphism $H \cong E_R(R/\mathfrak{m})$.

It therefore follows from Corollary 3.8 that, for each ideal $\mathfrak{b} \in \mathcal{I}(H)$, the complete reduced local ring $R/\mathfrak{b}$ satisfies the hypotheses of that corollary. This, together with Fedder's criterion for F-purity [5, Theorem 1.12] ensures a good supply of examples of rings to which the results of this paper apply.

3.10. **Example.** Suppose that $(R,\mathfrak{m})$ is local and complete, and that there exists an x-torsion-free left $R[x,f]$-module E which, as R-module, is isomorphic to $E_R(R/\mathfrak{m})$. Note that these hypotheses imply that R is reduced, by Corollary 3.8. Let $\mathfrak{p}$ be a minimal prime ideal of R.

Observe that $(0 :_R E) = 0$, and so $0 \in \mathcal{I}(E)$. Therefore, by [14, Theorem 3.6], we have $\mathfrak{p} \in \mathcal{I}(E)$, and it follows from Corollary 3.8 that the complete local domain $R/\mathfrak{p}$ satisfies the hypotheses of that corollary.

Another example is provided by M. Katzman's work in [10].

3.11. **Example.** Let $\mathbb{F}_2$ be the field of two elements, let T_1, T_2, T_3, T_4, T_5 be independent indeterminates, and let $R := \mathbb{F}_2[[T_1, T_2, T_3, T_4, T_5]]/\mathfrak{d}$, where $\mathfrak{d}$ is the ideal of

$\mathbb{F}_2[[T_1, T_2, T_3, T_4, T_5]]$ generated by the 2×2 minors of the matrix

$$\begin{pmatrix} T_1 & T_2 & T_2 & T_5 \\ T_4 & T_4 & T_3 & T_1 \end{pmatrix}.$$

Then R is Cohen–Macaulay but not Gorenstein, and it follows from the calculations reported by Katzman in [10, §9] that the injective envelope E of the simple R-module carries a structure of x-torsion-free left $R[x, f]$-module. Thus the conclusions of Corollary 3.8 apply to this R.

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